HEH 2305: OPERATIONS RESEARCH

**1. INTRODUCTION**

**Meaning and Scope**

**Definition of Operations Research**

Operations Research is a scientific methodology which is applied to the study of the operations of organizations (mostly large and complex) or activities to assess the overall implications of various alternative courses of action to provide an improved basis for management decisions.

OR is the application of scientific methods, techniques, and tools to problems involving the operations of systems to provide optimum solutions to these problems. OR is the use of scientific methods to provide criteria for decisions regarding machine systems involving repetitive operations.

**Characteristics of Operations Research**

1. It is inter-disciplinary. It borrows from different disciplines for developing new methods and procedures.
2. It is a continuous process.
3. It is objective is to find the best or optimal solution to the problem under consideration.
4. It increases the creative ability of a decision maker.
5. It is decision making science.
6. It uncovers new problems for study methods.
7. It examines fundamental relationships form a systems overview.
8. It uses scientific methodology.
9. It replaces management by personality.
10. It is for operations economy.

OR provides managers with quantitative basis for decision making. By employing a systematic study of a problem involving gathering data, building a mathematical model, experimenting with the model and predicting future operations; i.e. the mathematical and logical means of Operations Research provides the management with quantitative basis for decision making and enhances ability to make long-range plans and to solve everyday problems of running a business industry with greater efficiency, competence and confidence.

**2. ORIGIN OF OPERATIONS RESEARCH**

The need for effective allocation of the very limited (scarce) military resources of Britain during World War II led to the development of operations research. Many scientists including physicists, biologists, statisticians, mathematicians and psychologists applied a scientific approach to the many strategic and tactical problems which helped to win the “Air battle of Britain”, “Battle of North Atlantic”, and the Island Campaign in the Pacific. The success of this team of scientists in Britain encouraged U.S, Canada and France to start with such teams.

The name OR came directly from the context in which it was used and developed, viz., research on military operations. As the discipline of OR developed, many names like “operational analysis”, “systems analysis”, “cost benefit analysis”, “management analysis”, “decision science” etc. were assigned to it. The apparent success of the military team in the war prompted the industry sector that was experiencing complex decision problems, due to the increasing complexity and specialization in organizations, to use the formal tools of operations research.

**3. PHASES OF OPERATIONS RESEARCH**

The scientific method in OR study generally involves the following three phases:

1. **Judgment phase**

This phase consists of:

* 1. Determination of the operation
  2. Establishment of the objectives and values related to the operation
  3. Determination of the suitable measures of effectiveness
  4. Formulation of the problems relative to the objectives

1. **Research phase**

This phase utilizes:

* 1. Operations and data collection for better understanding of the problems
  2. Formulation of hypothesis and model
  3. Observation and experimentation to test the hypothesis on the basis of additional data
  4. Analysis of the available information and verification of the hypothesis using pre-established measures of effectiveness
  5. Predictions of various results from the hypothesis
  6. Generalisation of the various results and consideration of alternative methods

1. **Action phase**

It consists of making recommendations for the decision process by those who first posed the problem for consideration or by anyone in a position to make a decision, influencing the operation in which the problem occurred.

**Applications of Operations Research**

1. Determination of optimal product mix – optimum allocation of resources to competing products or activities, transportation schedules, rent allocation, assignment of personnel and machine, media selection, investment portfolio selection, blending of materials, energy, ecology (pollution) etc by linear programming.
2. Inventory control model aids in minimizing the sum of
3. Acquisition costs
4. Stock holding costs
5. Shortage costs for various stock items
6. PERT and CPM techniques are very useful in planning, analyzing, scheduling and controlling the progress and completion of large and one time projects.
7. Application of dynamic planning in areas such as planning, advertising expenditures, distributing sales efforts, and production scheduling.
8. Decision analysis has been applied to problems in controlling hurricanes, water pollution, machine, law, nuclear safety, space exploration, new product decisions, advertising expenditures, research and development.
9. Queuing theory has had application in solving problems concerned with traffic congestion, servicing machines subject to breakdown, determining the level of a service force, air traffic scheduling, design of dams, job shop scheduling, hospital operations, receipts and withdrawal counters in a commercial bank.
10. Replacement theory is extensively applied in funding the optimal replacement interval for:
11. Equipment that deteriorates with time
12. Equipment (e.g. bulbs) that perish suddenly, and
13. Staff replacement
14. The use of exponential smoothing for:
15. Preparation of monthly forecasts for stock-keeping items
16. Determination of optimal product inventory and work-force levels in production planning
17. Use of simulation in probabilistic marketing situations such as:

* to derive NPV/Internal rate of return distribution for the venture of introduction of a new product

**6. TECHNIQUES OF OPERATIONS RESEARCH**

1. **PROBABILITY**

* for prediction of the future of the business
* Probability concepts help to analyze the uncertainties and bring out necessary data with reasonable accuracy for decision making.

2 types of probabilities:

1. Objective probability: for which there is a definite historical evidence and common experience e.g. P(H) in tossing a coin is ½ assuming it is unbiased/fair coin
2. Subjective probability: where historical evidence is not available and the businessman has to guess the likelihood of various possible outcomes in a situation e.g. decision on the number of umbrellas and raincoats to be ordered by a salesman during the rainy season will depend on estimates or “educated guesses”.
3. **DECISION THEORY**

The basic elements in decision theory are:

* alternative courses of action
* various states of nature
* knowledge about the likelihood of occurrence of each state of nature
* net value (pay-off) to decision maker for each outcome
* decision maker’s objectives

The basic premise of decision theory is that the behaviour of the future is probabilistic and not deterministic. Various probabilities are assigned to the state of nature on the basis of available information or subjective judgment and the likely outcomes of the alternative courses of action are evaluated accordingly before a particular alternative is selected.

1. **LINEAR PROGRAMMING**

* Involves selection of an optimum combination of factors from a series of inter-related alternatives, each subjective to limitation.
* Involves development of linear equations to obtain the best solution for the allocation of the problem.
* Linear programming consists of:
  1. 59The simplex method – aims at maximizing or minimizing a given function subject to constraints in respect of each variable.
  2. The transportation problem – deals with the problems of matching the origins(stores, warehouses, factories) with the outlets (process, centre, market etc) at minimum cost of distribution and transportation
  3. The assignment problem – assigning a given number of agents each one to the same number of tasks so as to result in maximum efficiency or minimum cost.

1. **DYNAMIC PROGRAMMING**

* Deals with problems related to multi-period analysis and decisions.
* No standard mathematical formulation like in the case of linear programming.
* A problem is broken down into a series of problems in such a way that answers to the first sub problem can be used in deriving the solution to the next sub-problem and so forth, finally giving solution to the whole problem.

1. **SEQUENCING**

* Determination of a sequence in which given jobs should be performed to minimize total efforts – cost, time, mileage etc.
* Solves problem where effectiveness measures (in terms of cost, time, mileage etc) depends upon the sequence of performing given jobs.

1. **GAME THEORY**

* A mathematical theory applicable to competitive business problems.
* Deals with situations where 2 or more (finite) individuals are making decisions involving conflicting interest. The final decision depends upon the decisions of the parties concerned.
* The basic assumptions made are that every competing party will adopt the policy most unfavourable to us and therefore we are required to select the best position among the worst positions.
* In such situations, one’s favourable item is unfavourable to another.

1. **INVENTORY CONTROL AND MANAGEMENT**

* Inventory problems (models) are mainly concerned with inventory decisions which basically include:
  1. How much to order at one time
  2. When to order this quantity

1. **QUEUING THEORY**
2. **NETWORK ANALYSIS (PERT/CPM)**

* PERT – Programme Evaluation and Review Technique
* CPM – Critical Path Method
* PERT and CPM are management tools for planning and control of complex jobs involving a large number of activities.
* The objective of PERT and CPM techniques is to establish the total duration required for the completion of the project while incurring the optimum cost.

1. **SIMULATION**

* It is a manipulation of a model constructed from the formal statements of a mathematical representation in respect of logical relations between the elements in a structure or a system expressed in measurable terms.
* It is a process of designing an experiment which will duplicate or present as nearly as possible the real situation and then watching what does happen.
* It is especially appropriate where it is difficult to build a model for the real life situation mathematically or if at all it is modeled, it is difficult to solve the model analytically.

1. **REPLACEMENT THEORY**

* Determination of the time when items of a plant should be replaced due to depreciation.

1. **RELIABILITY THEORY**

* Quantification of the frequency of failures and development of an indicator of quality and dependability of a product.

1. **MARKET ANALYSIS**

* A method of analyzing the current movement of some variable in order to predict the future movement of the same variable.

**Advantages of Operations Research**

1. Better control of operations of various systems in big complex organizations. This spares management the cost and effort of continuous executive supervisions over routine decisions.
2. Better co-ordination of operations of systems. Maintains law and order to avoid chaos.
3. Better decisions – quantitative basis for decision making
4. Better systems – development of systems that are more effective and efficient.

**Disadvantages of Operations Research (OR)**

1. Magnitude of computation – OR seeks optimal solutions, taking all the factors into account. These factors are numerous and expressing them in quantity and establishing relationships among these requires huge calculations which require costly electronic computers.
2. Absence of quantification of some intangible factors – OR requires quantification of all elements. Some intangible factors such as human relations cannot be quantified. Thus they are excluded, yet they may be significant.
3. Distance between managers and OR – operations research requires specialized skills of mathematicians or statisticians who might not be aware of business problems. Similarly, a manager fails to understand the complex working to OR.

**Key Concepts**

1. **Analog model** – a model in which one physical property is used to represent another physical property.
2. **Decision theory models** – a class of OR models designed to select an optimal course of action from a set of alternative courses of action.
3. **Decision variables** – the unknowns that are to be determined by solving the model.
4. **Deterministic models** – a model in which the functional relationships and parameters are known with certainty.
5. **Iconic model** – a scaled physical representation of a real system.
6. **Operations research** – a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under control.

**7. LINEAR PROGRAMMING MODELS**

**Definition of Linear Programming**

Linear programming is a mathematical technique for determining the optimal allocation of resources among alternative uses of the resources to achieve a particular objective. The objective includes profit maximization or cost minimization.

**Advantages (utility) of L.P Approach**

1. Insight and perspective into problem solutions
2. Consideration of all possible solutions to the problem
3. Better and more successful decisions.
4. Better tools for adjusting to meet changing conditions
5. Optimal use (allocation) of resources
6. Insight and perspective into problem situations

**Limitations of L.P Approach**

1. Applicable only for linear expressions of objective functions and constraints.
2. Applicable only when coefficients in the objective function and the constraint are known and constant over period of study.
3. It may generate fractional valued solutions which may not be applicable or sensible.
4. It will fail to give a solution if management has conflicting multiple goals.
5. Large numbers of variables and complexity of their relationships.

**Application areas of L.P**

1. Industrial applications
   1. Product mix problem
   2. Production Scheduling e.g. arrangement of 20 jobs to be done on 5 machines
   3. Blending Problems – where a product can be made from a variety of available raw materials of various composition and prices
   4. Transportation problem
   5. Production distribution problems
2. Management applications
   1. Portfolio selection – selection of specific investments from among a wide variety of alternatives, to maximize returns or minimize risks.
   2. Financial mix strategy – selection of means for financial company projects, production operations etc. How much production is to be supported by internally generated funds and by external funds?
   3. Profit planning – to maximize profit margin from net investment in plant facilities and equipment, cash on hand and inventory.
   4. Media selection – in advertising field – effective media mix.
   5. Traveling salesmen problem.
   6. Determination of equitable salaries.
   7. Staffing problem
3. Miscellaneous applications
   1. Farm planning – allocation of limited resources (e.g. acreage, labour, water supply etc.) for revenue maximization.
   2. Airline routine – to determine the most economic pattern and timing for flights for the most efficient use of aircraft, crews and money.
   3. Diet problems – to determine the most economical diet for patients, feed ingredient combination to satisfy stated nutritional requirements at a minimum cost level.
   4. Administrative applications
      1. Optimal usage of resources like men, machine, materials i.e. optimal allocation of resources.
      2. Departmental staff requirements over a period of time.
      3. Work distribution among staff members according to their efficiency for optimum results.

**8. MATHEMATICAL MODELS FOR LINEAR PROGRAMMING**

2 main methods:

1. Graphical method
2. Simplex method

**GRAPHICAL METHOD**

Steps:

1. Write the objective function and all necessary constraints.
2. Graph the feasible region.
3. Determine the coordinates of each of the corner points.
4. Find the value of the objective function at each corner point.
5. For a bounded region, the solution is given by the corner point producing the optimum value of the objective function.
6. For an unbounded region, check that a solution actually exists. If it does, it will occur at a corner point.

Example 1

Maximize



Subject to



5s

10s

15s

20s

25s

30s

35s

5s

10

15

20

25







Corner points : (0,12) (0,0) (10,0) (10,6) (6,10)

Corresponding Z: 240 0 60 180 236

Optimal point (6,10)

Therefore, ,  and 

Example 2

Minimize:



Subject to:



5s

10s

15s

20s

25s

30s

35s

5

10







**SIMPLEX METHOD**

**Basic Concepts**

1. Standard form – a linear program in which all of the constraints are written as equalities.
2. Slack variable – a variable added to the left hand side of less than or equal to constraint to convert the constraint into an equality. Interpretation: the amount of unused resources.
3. Surplus variable – a variable subtracted from the left hand side of a greater than or equal to constraint to convert the constraint into an equality. Interpretation: the amount over and above the required minimum level.
4. Iteration – sequence of steps performed in moving form one basic feasible solution to another.
5. Simplex tableau – simplex table
6. Indicators – the numbers in the bottom row of the simplex table, which are from the objective function except for the 0 at the far right corner.
7. Pivot column – the column of simplex table with the most negative indicator.
8. Pivot row – the row with the smallest non-negative quotient.
9. Pivot or key element – the element at the intersection of pivot row and pivot column.

**Solving Maximization Problems**

**Standard Maximum Form**

A linear programming problem is in standard form if:

1. The objective function is to be maximized
2. All variables are non-negative (xi = 0)
3. All constraints involve 
4. The constants in the constraints are all non-negative 

Steps

1. Determine the objective function
2. Write all the necessary constraints
3. Convert each constraint into an equation by adding slack variables
4. Set up the initial simplex tableau
5. Locate the most negative indicator. Choose either if there are two.
6. Form the necessary quotients to find the pivot. Disregard any negative quotient or quotients with a zero-denominator. The smallest non-negative quotient gives the location of the pivot. If all quotients must be disregarded, no maximum solution exists. If two quotients are equally the smallest, let either determine the pivot.
7. Transform the tableau so that the pivot becomes 1 and all other numbers in that column becomes 0.
8. If all the indicators are all positive or 0, this is the final tableau. If not, go back to step 5 above and repeat the process until a tableau with no negative indicators is obtained.
9. Read the solution from this final tableau. The maximum value of the objective function is the number in the lower right corner of the final tableau.

Maximize



Subject to



Slack variables



Initial simplex tableau:



Most negative indicator is -10, pivot column is X2. 8/2 = 4. 16/2 = 8. 8 is least quotient. Pivot row is 1st row, pivot element = 2.









Maximum is 48 hours when X1=2, X2=3

Example 2

Maximize



Subject to



Convert the equalities by adding slack variables

Max 

Sub to











Solutions

, , , , , and . 

**Practice Questions**

1. Maximize



Subject to



Solutions 

1. Maximize



Subject to the constraints



Solutions 

1. The ABC manufacturing company can make 2 products, P1 and P2. Each of the products requires time on cutting machine and a finishing machine as shown below.

**Products**

**P1** **P2**

Cutting hrs (per unit) 2 1

Finishing hrs (per unit) 3 3

Profit (per unit) 6 4

Maximum sales (units per week) - 200

The number of cutting hours available per week is 390 and number of finishing hours available per week is 810. How much should be produced of each product in order to achieve maximum profit of the company.

Maximize (total profits)



Subject to



Where,  = No. of units of product P1 and  = No. of units of product P2.

Solution: 

Optimum mix leaves 5000 units of constraint no.3 and 2000 of constraints no. 1 used.

**9. MIXED CONSTRAINTS: PROBLEMS WITH  OR  CONSTRAINTS**

**Example 1**

Maximize



Subject to

Add slack or surplus variables to the constraints as needed



Phase 1

First simplex tableau



This tableau gives the solution:

 = 0  = 0  = - 60  = 120

But this is not a feasible solution since  is negative. All the variables in any feasible solution must be non-negative.

When a negative value of a variable appears, use row operations to transform the matrix until a solution is found in which all variables are non-negative. The difficulty is caused by the – 1 in row one of the matrix. Correct this by using row transformations to change a column that has non-zero entries (such as  or  columns) to one in which the first row is 1 and the other entries are 0. The choice of a column is arbitrary.

Let us choose the  column. Multiply the entries in the first row by  to get 1 in the top row of the column. Then use row transformations to get 0’s in the other rows of that column.



The solution, , ,  is feasible. The process of applying row transformations to get a feasible solution is called phase 1.

Phase 2

The simplex method is applied as usual to get the optimal solution in phase 2. The pivot is .



,  and 

Procedure/Steps:

1. Set up initial simplex tableau
2. Apply row transformations to get a feasible solution i.e. phase 1.
3. Use simplex method to get optimal solutions i.e. phase 2.

**Example 2**

Maximize



Subject to







Solution

, , , , , . Not a feasible solution because  is negative. Choose a column X2, 3rd row entry in column X2 is already 1. Using row transformation to get 0’s in the rest of the column gives:

Phase 1:



Phase 2:



Solutions,

, , , , , 

**Exercise**

1. Maximize



Subject to



Step 1



Step 2: Use 2nd column to complete phase 1 of the solution and give the feasible solution of the results.



Step 3: Use simplex method to solve.

1. An animal feed company must produce 200kg of a mixture consisting of ingredient  and  daily.  costs $ 3 per kg and  $ 8 per kg. No more than 80 kg of  can be used, and at least 60 kg of  must be used. Find how much of each ingredient should be used if the company wants to minimize cost.

Minimize (total cost)



Subject to constraints



where  = no. of kgs of ingredient  and  = no. kgs of ingredient 

, , 

Solutions:

, , 

**10. MINIMIZATION PROBLEM**

Standard minimum form:

A linear programming problem is in standard minimum form if:

1. The objective function is to be minimized
2. All the variables are non-negative
3. All the constraints involve 
4. The constants in the constraints are all non-negative. Standard minimum problems can be solved with the method of surplus variables presented above. To solve a minimization problem, first observe that the minimum of an objective function is the same number as the maximum of the negative of the function.

**Procedure: Solving nonstandard problems**

1. If necessary, convert the problem to a maximum problem
2. Add slack variables and subtract surplus variables as needed.
3. Write the initial simplex tableau.
4. If the solution from this tableau is not feasible, use row transformations to get a feasible solution (phase 1).
5. After a feasible solution is reached, solve by the simplex method (phase 2)

Example

Minimize



Subject to



and



Step 1: Change this to a maximization problem by letting Z equal the negative of the objective function. Z = -10. Then find the maximum value of Z.

 The problem can now be stated as follows:

Maximize



Subject to



and



Step 2: For phase 1 of the solution, subtract surplus variables and set up the first tableau.

Maximize 

Subject to 

Step 3: Initial simplex tableau



Step 4: Row transformations

The solutions, ,  and  contains negative numbers. Row transformations must be used to get a tableau with a feasible solution. Let us use the  column since it already has a 1 in the first row. Begin by getting a 0 in the second row, and then a 0 in the third row.



Step 5: Solution by simplex method (phase 2)

The above tableau has a feasible solution. In phase 2, complete the solution as usual by the simplex method. The pivot is -5



The solution is, ,  and . This solution is feasible, and the tableau has no negative indicators. Since  and Z = -W, then  is the minimum value, which is obtained when  and.

**Exercise**

1. Minimize



Subject to



and



Solution

, ,  and , 

1. A chemical manufacturer processes two chemicals, Arkon and Zenon, in varying proportions to produce three products A, B, and C. He wishes to produce at least 150 units of A, 200 units of B and 60 units of C. Each ton of Arkon yields 3 of A, 5 of B and 3 of C. Each ton of Zenon yields 5 of A, 5 of B and 1 of C. If Arkon costs $ 40 per ton and Zenon $ 50 per ton, advise the manufacture how to minimize his cost.

Solution

Minimize



Subject to



where  = no. of tons of Arkon and  = no. of tons of Zenon

Ans: Purchase 25 tons of Arkon

Purchase 15 tons of Zenon.

Total cost 

1. Sam, who is dieting requires tow food supplements I, and II. He can get these supplements from two different products, A and B as shown in the table.

|  |  |  |
| --- | --- | --- |
|  | Supplement  (Grams per serving) | |
|  | I | II |
| A | 3 | 2 |
| B | 2 | 4 |

Sam’s physician has recommended that he include at least 15 grams of supplement I but no more than 12 grams of supplement II in his daily diet. If product A costs $ 25 per serving and product B costs $ 40 per serving, how can he satisfy his requirements most economically?

**11. DUALITY THEORY**

Duality is related to a connection that exists between standard maximum and standard minimum problem. Any solution of a standard maximum problem produces the solution of an associated standard minimum problem, and vice-versa. Each of these associated problems is called the dual of the other.

**Theorem of Duality**

The objective function W of a minimizing linear programming problem takes on a minimum value if and only if the objective function Z of the corresponding dual maximizing problem takes on a maximum value. The maximum value of Z equals the minimum value of W.

**Solving minimum problems with Duals**

1. Find the dual standard maximum problem
2. Solve the maximum problem using the simplex method
3. The minimum value of the objective function W is the maximum value of the objective function Z.
4. The optimum solution is given by the entries in the bottom row of the columns corresponding to the slack variables.

Example

Minimize



Subject to



Step 1: Get the dual maximizing problem; use the given information to write the matrix



Transpose to get the following matrix for the dual problem



Write the dual problem from this matrix as follows:

Maximize



Subject to



Step 2: Solve this standard maximizing problem using simplex method.

Slack variables





Simplex method gives the following final tableau:



The last row of this final tableau shows that the solution of the given standard minimum problem is as follows:

The minimum value of W is.  and. Minimum value of W,  is the same as maximum value of Z.

**Interpretation of primal-dual optimum solution**

1. Slack/surplus variables in the dual problem correspond to the primal basic variables in the optimum solution.
2. The values of the elements in the index row corresponding to the columns of the slack/surplus variables with changed sign directly gives the optimum values of the basic primal variables.
3. Values for the slack variables of the primal are given by the index row under the non-basic variables of the dual solution with changed sign.
4. The value of the objective function is same for primal and dual problem.

**Exercise**

Minimize



Subject to



Solution:  for a minimum of 40

**12. SENSITIVITY ANALYSIS**

Also referred to as post-optimality analysis. It is usually undertaken to explore the effect of changes in the LP parameters on the optimum solution. Sensitivity analysis is concerned with the extent of sensitivity of the optimum solution to an LP for change in one or more of:

1. The profit or cost co-efficients of the objective function
2. The LHS coefficients of the variables in the constraints
3. The RHS quantities of the constraints

Sensitivity analysis is the process of investigating to what extent the numerical parameters of a LP model can be changed before the optimum solution for a given set of parameters of a LP model can be changed, before the optimum solution for a given set of parameters is disturbed. Such post optimality analysis can be of the following types.

**13. TRANSPORTATION MODELS**

Transportation models deal with the transportation of a product manufactured at different plants or factories (supply origins) to a number of different warehouses (demand destinations). The objective is to satisfy the destination requirements within the plants capacity constraints at the minimum transportation cost.

Transportation models typically arise in situations involving physical movement of goods from plants to warehouses, warehouses to wholesalers, wholesalers to retailers, and retailers to customers.

Solution of the transportation models requires the determination of how many units should be transported from each supply origin to each demands destination in order to satisfy all the destination demands while minimizing the total associated cost of transportation.

**Illustration:**

Assume that a manufacturer has three plants, P1, P2 and P3 producing the same product, which is transported from them to three warehouses W1, W2 and W3. Each plant has a limited supply (capacity), and each warehouse has specific demand. Each plant can transport to each warehouse but the transportation costs vary for different combinations. The problem is to determine the quantity each plant should transport to each warehouse in order to minimize total transportation costs.

P1

P2

P3

W1

W2

W3

Supply

S1

S2

S3

D1

Demand

D2

D3

**LINEAR PROGRAMMING FORMULATION OF TRANSPORTATION PROBLEM**

Let





Objective function is to minimize total transportation costs.

The LP objective function:

Minimize



Subject to:

Supply constraints



Demand constraints



And. Assumed that total supply available at the plants will exactly satisfy the demand required at the destinations, i.e.



In general a transportation problem with *m* factories (capacity centres or source of supply) and *n* warehouses (requirement centres) can be expressed in tabular form as follows:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Factory  (Origins) | Warehouse destinations | | | | | | Supply  (Availability) |
|  | W1 | W2 |  | Wj |  | Wn |  |
| F1 |  |  |  |  |  |  | S1 |
| F2 |  |  |  |  |  |  | S2 |
|  |  |  |  |  |  |  |  |
| Fi |  |  |  |  |  |  | Si |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  | Sm |
| Demand/ Requirement | d1 | d2 |  | d j |  | d n |  |

The problem of the company is to distribute the available product to different warehouses in such a way so as to minimize the total transportation cost for all possible factory-warehouse shipping patterns.









= *supply or capacity in units at origin i*

= *demand in units at destination j*

1. The objective is to determine xij that would minimize the total transportation cost:





Subject to the linear constraints:

1. Total supply form the ith origin to all the destinations is equal to total quantity produced at the ith origin, i.e.



1. Total quantity transported at the jth destination from various origin should be equal to the quantity required at the jth destination, i.e.





1. The general mathematical model may be given as follows:



*Subject to the constraints*

.





For a feasible solution to exist, it is necessary that total capacity equals total requirement, i.e.



**Feasible solution**

A set of non-negative values xij, *i = 1,2,…,m; j = 1,2,…,n* that satisfies (ii), (iii) and (iv) is called a feasible solution to the transportation problem.

**Basic feasible solution**

It is an initial feasible solution with an allocation *(m+n-1)* number of variables,

*xij = 1,2,…,m; ; j = 1,2,…,n*

**Optimum solution**

It is a feasible solution (not necessarily basic) that minimizes the total transportation cost.

Illustration



Subject to

1. Capacity constraints



1. Requirement constraints



and where  represents the number of unites of the product shipped from the ith production centre (*i=1,2,3,4)* to the jth selling centre *(j=1,2,3)*. The value of each  will be positive for a cell, this means that no quantity is shipped between the production centre and selling centre in question.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| From/To | W1 | W2 | W3 | W4 | Capacity  (supply) |
| F1 | 21 | 16 | 25 | 13 | 11 |
| F2 | 17 | 18 | 14 | 23 | 13 |
| F3 | 32 | 27 | 18 | 41 | 19 |
| Requirement  (demand) | 6 | 10 | 12 | 15 | 43 |

The above LP problem involves decision variables and 3 + 4 =7 constraints.

**SOLUTION PROCEDURE FOR TRANSPORTATION PROBLEMS**

Transportation method: procedure

1. Define the objective function to be minimized with the constraints imposed on the problem.
2. Set up the transportation table with *m* rows representing the sources (plants, factories etc.) and *n* columns representing the destinations (warehouses, stores, markets etc).
3. Develop an initial feasible solution to the problem
4. Examine whether the initial solution is feasible or not. The solution is said to be feasible if the solution has allocations in *(m+n-1)* cells with independent positions. The positions are said to be independent if it is not possible to alter any individual allocation without either changing the positions of allocations or violating the supply and demand constraints. The cells having allocations are known as occupied cells and the remaining cells are known as empty (or unoccupied) cells.
5. Test whether the solution obtained in step 4 is optimum or not. This is done by computing opportunity costs associated with the empty cells. Opportunity cost of an empty cell mean what will be the cost reduction, if this particular cell is included in the solution.
6. If the solution is non-optimum, modify the shipping schedule by including that empty cell whose inclusion in the programme results in the largest savings.
7. Repeat steps 5 and 6 until an optimum solution is obtained.

**Applications**

1. To minimize shipping costs from factories to warehouses or from warehouses to retail outlets.
2. To determine lowest cost location for new factory, warehouse or sales office.
3. To determine minimum cost production schedule that satisfies the firm’s demand and production limitations (called ‘production smoothing’).

**FINDING AN INITIAL FEASIBLE SOLUTION**

An initial basic feasible solution can be constructed by selection the *(m+n-1)* basic variable (allocations)  one at a time after which a value is assigned to that variable so as to satisfy a linear constraint.

3 methods:

1. The north-west corner rule
2. Lowest cost entry method
3. Vogel’s approximation method
4. **North-West corner method (NWCM)**
5. Select the north west (upper left hand) corner cell of the transportation table and allocate as many units as possible equal to the minimum between available supply and demand requirement i.e. min (s1, d1)
6. Adjust the supply and demand numbers in the respective rows and columns allocations.
7. If:
   1. The supply for the first row is exhausted, then move down to the first cell in the second row and first column and go to step 2.
   2. If the demand for the first column is satisfied then move horizontally to the next cell in the second column and first row and go to step 2.
8. If for any cell, supply equals demand, then the next allocation can be made in cell either in the next row or column.
9. Continue the procedure until the total available quantity is fully allocated to the cells as required.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| From/To | W1 | W2 | W3 | W4 | Capacity  (supply) |
| F1  6 | 21  5 | 16 | 25 | 13 | 11 |
| F2 | 17  5 | 18  8 | 14 | 23 | 13 |
| F3 | 32 | 27  4 | 18  15 | 41 | 19 |
| Requirement  (demand) | 6 | 10 | 12 | 15 | 43 |

Solution consists of six cells





1. **Least Cost Method (LCM)**
   1. Select the cell with the lowest transportation cost among all the rows or columns of the transportation table.
   2. If the minimum cost is not unique, then select arbitrarily any cell with this minimum cost
2. Allocate as many units as possible to the cell determined in step (i) and eliminate that row (column) in which either supply is exhausted or demand is satisfied.
3. Repeat steps (i) and (ii) for the reduced table until the entire supply at different factories is exhausted to satisfy the demand at different warehouses.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| From/To | W1 | W2 | W3 | W4 | Capacity  (supply) |
| F1 | 21 | 16 | 25  11 | 13 | 11 |
| F2  1 | 17 | 18  12 | 14 | 23 | 13 |
| F3  5 | 32  10 | 27 | 18  4 | 41 | 19 |
| Requirement  (demand) | 6 | 10 | 12 | 15 | 43 |



1. **Vogel’s Approximation Method (VAM)**

More preferred to the other two methods because the initial basic feasible solution obtained is either optimum or very close to the optimum solution.

1. Compute a penalty for each row and column in the transportation table. The penalty for a given row and column is merely the difference between the smallest cost and the next smallest cost in that particular row or column.
2. Identify the row or column with the largest penalty. In this identified row or column, choose the cell which has the smallest cost and allocate the maximum possible quantity to the lowest cost cell in that row or column so as to exhaust either the supply at a particular source or satisfy demand at a warehouse.

If a tie occurs in the penalties, select that row/column which has the minimum cost. If there is a tie in the penalties, select that row/column which has minimum cost. If there is a tie in the minimum cost also, select that row/column which will have maximum possible assignments.

1. Reduce the row supply or column demand by the amount assigned to the cell.
2. If the row supply is now zero, eliminate the row, if the column demand is now zero, eliminate the column. If both the row supply and the column demand are zero, eliminate both the row and column.
3. Recompute the row and column difference for the reduced transportation table, omitting rows or columns crossed out in the preceding step
4. Repeat the above procedure until the entire supply at factories are exhausted to satisfy demand at different warehouses.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| From/To | W1 | W2 | W3 | W4 | Capacity  (supply) | Column  penalty |
| F1 | 21 | 16 | 25 | 13 | 11 | 3 |
| F2 | 17 | 18 | 14 | 23 | 13 | 3 |
| F3 | 32 | 27 | 18 | 41 | 19 | 9 |
| Requirement  (demand) | 6 | 10 | 12 | 15 | 43 |  |
| Row Penalty | 4 | 2 | 4 | 10 |  |  |

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| From/To | W1 | W2 | W3 | W4 | Capacity  (supply) | Column  Penalty |
| F2 | 17 | 18 | 14 | 23 | 13 | 3 |
| F3 | 32 | 27 | 18 | 41 | 19 | 9 |
| Requirement (demand) | 6 | 10 | 12 | 4 | 32 |  |
| Row Penalty | 15 | 9 | 4 | 18 |  |  |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| From/To | W1 | W2 | W3 | Capacity  (supply) | Column penalty |
| F2 | 17 | 18 | 14 | 9 | 3 |
| F3 | 32 | 27 | 18 | 19 | 9 |
| Requirement (demand) | 6 | 10 | 12 | 28 |  |
| Row Penalty | 15 | 9 | 4 |  |  |

Arbitrarily select column for W2 or row for F3 since both equal maximum difference. Select row for F3.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| From/To | W2 | W3 | Capacity  (supply) | Column penalty |
| F2 | 18 | 14 | 3 | 3 |
| F3 | 27 | 18 | 19 | 9 |
| Requirement (demand) | 10 | 12 | 22 |  |
| Row Penalty | 9 | 4 |  |  |

The next step brings in the cell (2,2) assignment x22 = 3 followed by the entry x32 = 7 unites which certainly satisfies the demand of warehouse W2. The initial feasible solution thus obtained is:





|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| From/To | W1 | W2 | W3 | W4 | Capacity  (supply) |
| F1 | 21 | 16 | 25  11 | 13 | 11 |
| F2  6 | 17  3 | 18 | 14  4 | 23 | 13 |
| F3 | 32  7 | 27  12 | 18 | 41 | 19 |
| Requirement  (demand) | 6 | 10 | 12 | 15 | 43 |

Note: TC is considerably less than the cost associated with the initial solutions obtained by the other 2 methods.

**14. ASSIGNMENT MODELS**

Assignment problem deals in allocation of various resources (items) to various activities (receivers) on one to one basis in such a way that the resultant effectiveness is optimized. Possible applications include assignment of available sales-force to different regions; vehicles to routes; products to factories; contracts to bidders; machines to jobs; development engineers to several construction sites and etc.

With information about the number of assignment *i,* *(i=1,2,3,…,n)* performing the same number of jobs *j* *(j=1,2,3…,n)* and the pay off measure Cij available for each assignment, the objective is to determine the strategy that minimizes the total cost or maximizes the total utility.

*n* resources are to be assigned to *n* activities such that each resource is allocated to each activity and each activity is performed by one resource only. The allocation is to be done in such a way so as to maximize the resultant effectiveness.

The assignment problem can be represented in the form of ** matrix ** known as cost or effectiveness; where** represents assignment of the jth person to the ith job and ** is the cost associated with assigning the ith facility(person) to the jth job.

It is assumed that:

*=1,* if ith person is assigned to the jth job and

*=0*, otherwise.

Mathematical formulation of the assignment problem is given as



subject to the constraints:

1. Each person should be assigned to one and only one job, i.e.

**

1. Each job must be assigned to one and only one person, i.e.

**

1. **

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Persons/Jobs | J1 | J2 |  | Jn | Supply |
| P1 | c11  x11 | c12  x12 |  | c1n  x1n | 1 |
| P2 | c21  x21 | c22  x22 |  | c1n  x2n | 1 |
|  |  |  |  |  |  |
| P3 | cn1  xn1 | cn2  xn2 |  | cnn  xnn | 1 |
| Requirement | 1 | 1 |  | 1 |  |

**Hungarian Method of Assignment Problem (Minimization Case)**

This method is based on the following important properties:

1. In assignment problem, if a constant quantity is added or subtracted from every element of any row or column in the given cost matrix, an assignment that minimizes the total cost in one matrix also minimizes the total cost in the other.
2. In assignment problem, a solution having zero total cost is an optimum solution.

**Procedure for Hungarian Method**

1. In a given matrix, subtract the smallest element in each row from every element of that row.
2. In the reduced matrix obtained from step 1, subtract the smallest element in each column from every element of that column.
3. Make the assignment for the reduced matrix obtained from steps 1 and 2 in the following ways:
4. Examine the rows successively until a row with exactly one unmarked zero is found. Enclose this zero in a box () as an assignment will be made there and cross (X) all other zeros appearing in the corresponding column as they will not be considered for future assignment. Proceed in this way until all the rows have been examined.
5. Examine the columns successively until a column with exactly one unmarked zero found. Make an assignment to this single zero by putting a square () around it and cross out (X) all other zeros appearing in the corresponding row as they will not be used to make any other assignment in that row. Proceed in this manner until all columns have been examined.
6. Repeat the operations (i) and (ii) successively until one of the following situations arises:
   1. All the zeros in rows/columns are either marked () or crossed (X) and there is exactly one assignment in each row and in each column. In such a case optimum assignment policy for the given problem is obtained.
7. There may be some row (or column) without assignment, i.e. the total number of marked zeros is less than the order of the matrix. In such a case, proceed to step 4.
8. Draw the minimum number of vertical and horizontal lines necessary to cover all the zeros in the reduced matrix obtained from step 3 by adopting the following procedure:
9. Mark () all rows that do not have assignments.
10. Mark () all columns (not already marked) which have zeros in the marked rows [step 4(ii)]
11. Mark () all rows (not already marked) that have assignments in marked columns [step 4(iii)].
12. Repeat steps 4 (ii) and (iii) until no more rows or columns can be marked.
13. Draw straight lines through all unmarked rows and marked columns.
14. If the number of lines drawn [step 4 (iii)] are equal to the number of rows or columns, then it is an optimum solution, otherwise go to step 6.
15. Select the smallest element among all the uncovered elements. Subtract this smallest element from all uncovered elements and add it to the element which lies at the intersection of two lines. Thus we obtain another reduced matrix for fresh assignments.
16. Go to step 3 and repeat the procedure until the number of assignments become equal to the number of rows and columns. In such a case, we shall observe that row/column has an assignment. Thus the current solution is an optimum solution.

**Steps in Assignment Problem**

1. Set up cost table for the problem
2. Find the opportunity cost
3. Subtract smallest number in each row from every number in that row, then
4. Subtract smallest number in each column from every number in that column.
5. Test opportunity cost table to see if optimum assignments are possible by drawing the minimum possible lines on columns and/or rows such that all zeroes are covered.

If optimum: No. of lines = no. of rows or columns

Optimum solutions at zero locations systematically make final assignments.

1. Check each row and column for a unique zero, and make the first assignment at the intersection.
2. Eliminate that row and column and search for another unique zero. Make that assignment and proceed in a like manner.

If not optimum: No. of Lines < No. of Rows or Columns

1. Revise opportunity cost table in 2 steps:
2. Subtract the smallest number not covered by a line from itself and every other uncovered number.
3. Add this number at every intersection of any two lines.

**Example 1**

A computer centre has got 4 expert programmers. The centre needs 4 application programmes to be developed. The head of the computer centre, after studying carefully the programmes to be developed, estimates the computer time in minutes required by the respective experts to develop the application programmes as follows.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **Programmes** | | | |
| **Programmers** | **A** | **B** | **C** | **D** |
| **1** | 120 | 100 | 80 | 90 |
| **2** | 80 | 90 | 110 | 70 |
| **3** | 100 | 140 | 120 | 110 |
| **4** | 90 | 90 | 80 | 90 |

**Step 1:** Subtract the smallest element of each row from every element of the corresponding row,

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **Programmes** | | | |
| **Programmers** | **A** | **B** | **C** | **D** |
| **1** | 40 | 20 | 0 | 10 |
| **2** | 10 | 20 | 40 | 0 |
| **3** | 0 | 40 | 20 | 10 |
| **4** | 10 | 10 | 0 | 10 |

**Step 2:** Subtract the smallest element of each column from every element in that column.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **Programmes** | | | |
| **Programmers** | **A** | **B** | **C** | **D** |
| **1** | 40 | 10 | 0 | 10 |
| **2** | 10 | 10 | 40 | 0 |
| **3** | 0 | 30 | 20 | 10 |
| **4** | 10 | 0 | 0 | 10 |

**Step 3:** Draw the minimum possible number of horizontal and vertical lines so as to cover all the zeros. Since the number of lines (= 4) is equal to the order of the given matrix, an optimum assignment has been attained. Make the zero assignment as follows:

1. Starting with row 1 of the matrix above, examine rows one by one until a row containing exactly a single zero element is found. Make an assignment indicated by () to that cell. Then cross all other zeros in the column in which the assignment was made. This eliminates the possibility of making further assignments in that column. Do likewise for the rows.
2. Starting with column 1, examine columns until a column containing exactly one remaining zero is found. Make an assignment in that position and cross other zeros in the row in which the assignment was made.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **Programmes** | | | |
| **Programmers** | **A** | **B** | **C** | **D** |
| **1** | 40 | 10 | 0 | 10 |
| **2** | 10 | 10 | 40 | 0 |
| **3** | 0 | 30 | 20 | 10 |
| **4** | 10 | 0 | 0 | 10 |

1. Continue with these successive operations on rows and columns until all zeros have been either assigned or crossed out. Absence of any more unmarked zeros indicates an optimum assignment schedule – as the table above. Then compute the minimum computer – time (in minutes) as follows:

**Example 2**

A company is producing a single product and is selling it through fire agencies situated in different cities. All of a sudden there is a demand for the product in another five cities not having any agency of the company. The Company is faced with the problem of deciding on how to assign the existing agencies to dispatch the product to needy cities in such a way that the traveling distances is minimized. The distance (in kms) between the surplus and deficit cities are given in the following distance – matrix.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Surplus cities/Deficit cities** | **Programmes** | | | | |
| **Programmers** | **I** | **II** | **III** | **IV** | **V** |
| A | 160 | 130 | 175 | 190 | 200 |
| B | 135 | 120 | 130 | 160 | 175 |
| C | 140 | 110 | 155 | 170 | 185 |
| D | 50 | 50 | 80 | 80 | 110 |
| E | 55 | 35 | 70 | 80 | 105 |

Determine the optimum assignment schedule.

**Step 1**

Solution: Subtracting the smallest element of each row from every element of the corresponding row, we get the adjoining reduced matrix.

Table 1

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Surplus cities/Deficit cities** | **Programmes** | | | | |
| **Programmers** | **I** | **II** | **III** | **IV** | **V** |
| A | 30 | 0 | 45 | 60 | 70 |
| B | 15 | 0 | 10 | 40 | 55 |
| C | 30 | 0 | 45 | 60 | 75 |
| D | 0 | 0 | 30 | 30 | 60 |
| E | 20 | 0 | 35 | 45 | 70 |

**Step 2**

Subtracting the smallest demand of each column from every element of the corresponding column to get the following reduced matrix:

Table 2

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Surplus cities/Deficit cities** | **Programmes** | | | | |
| **Programmers** | **I** | **II** | **III** | **IV** | **V** |
| A | 30 | 0 | 35 | 30 | 15 |
| B | 15 | 0 | 0 | 10 | 0 |
| C | 30 | 0 | 35 | 30 | 20 |
| D | 0 | 0 | 20 | 0 | 5 |
| E | 20 | 0 | 25 | 15 | 15 |

**Step 3**

Row 1 has a single zero in column 2. Make assignment by putting square around it and delete the other zeros in column 2 by marking X.

Table 3

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Surplus cities/Deficit cities** | **Programmes** | | | | |
| **Programmers** | **I** | **II** | **III** | **IV** | **V** |
| A | 30 | 0 | 35 | 30 | 15 |
| B | 15 | 0 | 0 | 10 | 0 |
| C | 30 | 0 | 35 | 30 | 20 |
| D | 0 | 0 | 20 | 0 | 5 |
| E | 20 | 0 | 25 | 15 | 15 |

* Make an assignment by putting square (squaring 0) in column 1 which has a single demand in row 4 and cross the other zero which is not yet crossed.
* Column 3 has a single zero in row 2 thus, make an assignment and delete the other zero which is uncrossed.

**Note:** There are no remaining zeros, and row 3, row 5, column 4, and column 5 each has no assignment. Thus, the optimum solution is not reached at this stage. Proceed to the following important steps.

**Step 4**

Draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once. The following systematic procedure may help to draw the minimum set of lines:

1. For simplicity, first make table 4 again

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Surplus cities/Deficit cities** | **Programmes** | | | | |
| **Programmers** | **I** | **II** | **III** | **IV** | **V** |
| A | 30 | 0 | 35 | 30 | 15 | 4 |
| B | 15 | 0 | 0 | 10 | 0 | L2 |
| C | 30 | 0 | 35 | 30 | 20 | 1 |
| D | 0 | 0 | 20 | 0 | 5 | L3 |
| E | 20 | 0 | 25 | 15 | 15 | 2 |

L1

1. Secondly, mark () row 3 and row 5 in which there is no assignment.
2. Then mark () column 2 which have zeros in marked row 3 and 5.
3. Next, mark () row 1 because this row contains assignment in marked column 2. No further rows or columns will be required to mark during this procedure.
4. Draw through the required lines as follows:
   1. Draw line (L1) through marked column 2.
   2. Next, draw lines (L2 and L3) through unmarked rows (2 and 4) both having largest number (2) of uncovered zeros. Since no zero is left uncovered, the required lines will be L1, L2, and L3.

**Step 5**

1. Select the smallest element (15) from all the elements of the matrix not covered by lines and add to every element that lies at the intersection of the lines L1, L2, and L3 and leaving the remaining elements unchanged. Matrix table 5 is obtained.

Table 5

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Surplus cities/Deficit cities** | **Programmes** | | | | |
| **Programmers** | **I** | **II** | **III** | **IV** | **V** |
| A | 15 | 0 | 20 | 15 | 0 |
| B | 15 | 15 | 0 | 10 | 0 |
| C | 15 | 0 | 20 | 15 | 5 |
| D | 0 | 15 | 20 | 0 | 5 |
| E | 5 | 0 | 10 | 0 | 0 |

**Step 6**

Again repeat step 3 to reach the final matrix (table 6)

Table 6

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Surplus cities/Deficit cities** | **Programmes** | | | | |
| **Programmers** | **I** | **II** | **III** | **IV** | **V** |
| A | 15 | 0 | 20 | 15 | 0 |
| B | 15 | 15 | 0 | 10 | 0 |
| C | 15 | 0 | 20 | 15 | 5 |
| D | 0 | 15 | 20 | 0 | 5 |
| E | 5 | 0 | 10 | 0 | 0 |

**Note**

In table 6, there are no remaining zeros and every row and column has an assignment. Since no two assignments are in the same column (they cannot be if the procedure has correctly been followed) the zero assignment is the required optimum solution. From the original matrix, the optimum assignment schedule is therefore:

|  |  |
| --- | --- |
| **Route** | **Distance**  **(Kilometres)** |
| *A – V* | *200* |
| *B – III* | *130* |
| *C – II* | *110* |
| *D – I* | *50* |
| *E – IV* | *80* |
| ***Minimum distance***  ***travelled*** | ***570 Kilometres*** |

**Exercise**

1. A project work consists of four major jobs for which four contractors have submitted tenders. The tender amounts quoted in thousands of Ksh. are given in the matrix as:

Jobs

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | J1 | J2 | J3 | J4 |
| C1 | 15 | 29 | 35 | 20 |
| C2  Contractors | 21 | 27 | 33 | 17 |
| C3 | 17 | 25 | 37 | 15 |
| C4 | 14 | 31 | 39 | 21 |

Find the assignment which minimizes total cost of the project. Each contractor has to be assigned one job.

*Answers: C1 – J2; C2 – J3; C3 – J4; C4 – J4*

*Minimum cost = 91*

1. A company employs service engineers based at various locations throughout the country to service and repair their equipment installed in customers’ premises. Four requests for service have been received and the company finds that four engineers are available. The distance each of the engineers is from the various customers is given in the following table and the company wishes to assign engineers to customers to minimize the total distance to be traveled.

Customers

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | W | X | Y | Z |
| Alf | 25 | 18 | 23 | 14 |
| Bill  Engineers | 38 | 15 | 53 | 23 |
| Charlie | 15 | 17 | 41 | 30 |
| Dave | 26 | 28 | 36 | 29 |

*Solution:*

*Total mileage of the final assignment:*

|  |  |
| --- | --- |
|  | ***Mileage*** |
| *A to Z* | *14* |
| *B to X* | *15* |
| *C to W* | *15* |
| *D to Y* | *36* |
|  | ***80 miles*** |

**15. NETWORK ANALYSIS**

It refers to a family of related techniques that help management to plan and control projects. These techniques show interrelationships of various chores/tasks which make up the overall projects and clearly identify the critical paths of the project. They are most useful in complex large organizations or big projects with restrictions or constraints.

**Objectives of network analysis**

1. Planning, scheduling and control
2. Simplified framework of the interconnectivity of various activities constituting a project or a programme. Done to show the technology interdependence of various activities.
3. Minimization of the total cost, total time, and other resources.
4. Maximization of efficiency/effectiveness, returns, etc.

**MANAGEMENT APPLICATIONS OF NETWORK ANALYSIS**

Network analysis is the organized application of systematic reasoning for planning, scheduling, monitoring and controlling practical situations where many separate tasks (which make up a whole project) can happen simultaneously or consecutively.

**Applications include:**

1. Building and construction – construction of buildings, bridges, factories, highways, stadiums.
2. Budgeting and auditing procedures.
3. Assembly line scheduling
4. Missile development
5. Installation of complex equipment e.g. computers, large machines.
6. Planning.
7. Advertising programmes for development and launching of new products.
8. Finding the best traffic flow pattern in a city.
9. Research and development.

**Main concepts of network analysis***:*

1. **Activity –** a task or job which takes time and resources. Denoted by an arrow (**).**
2. **Predecessor activity –** an activity that must be completed immediately prior to the start of another activity.
3. **Successor activity –** activity that succeeds other activities.
4. **Concurrent activities –** activities that can be accomplished together.
5. **Critical activity –** an activity whose delay causes delay in the entire project.
6. **Dummy activity –** an activity that does not consume time or resources. It is used to show logical dependencies between activities. Denoted by a broken arrow (**).**
7. **Event –** a point in time that indicates the start or finish of an activity. It is denoted by a circle (O), or a node.

Activity

Start Event Finish event

There are 3 types of events:

1. Merge event – formed when more than one activity come and join together in an event.
2. Burst event - formed when more than one activity leaves an event.
3. Merge and burst event – a combination of the two. When more than one activity come and join together and more than one activity leaves an event.
4. **Network –** a combination of activities and events in a logical sequence according to the rules of drawing networks.

**Rules for construction of network diagrams**

1. Each activity is represented by one and only one arrow in the network.
2. No two activities can be identified by the same beginning and end event. In such cases a dummy activity is used.
3. An activity can only be undertaken if all activities preceding it are completed.
4. The flow of the network diagram should be from left to right.
5. Arrows should be straight, not curved or bent, and should not cross each other unless it is inevitable.
6. The length of arrows is of no significance.
7. Angles between the arrows should be as large as possible.
8. Each activity must have a unique tail and head event.

Activity

Tail event Head event

1. A complete network should have only one point of entry (i.e. start event) and only one point of exit (finish event).
2. Generally no even can be numbered until all preceding events have been numbered. The number at the head of the arrow is always greater than the number at its tail.

**Identification of activities**

There are three main methods of identification:

1. a short description of the task/activity
2. alphabetical or numerical code
3. tail and head event numbers

**Illustrations**

1. Activity B cannot start until A is completed. (B depends on A or B is preceded by A or B succeeds A).

A

B

1. Activities B and C cannot start until activity A is completed i.e. B and C depends on A.

A

B

C

1. Activity C cannot start until both A and B are completed

A

B

C

1. Both A and B must be finished before either C or D can start.

A

B

C

D

1. Activity A and B must be completed before activity C can start. Only activity B must be completed before activity D can start.

A

C

B

D

1. **Activity Preceding activity**

A: Buying orbit -

B: Removing the wrapper A

C: Putting the orbit in the mouth B

D: Chewing the orbit C

D

A

B

C

1. **Activity Preceding activity**

A -

C A

B -

D A, B

A

C

B

D

**Common errors in Network Diagrams**

1. Dangling – disconnection of an activity before completion of all activities in the project network diagramming.
2. Looping or cycling error – this is having a series of activities which lead back to the same event.
3. Redundancy – providing unnecessary dummy activity in a network diagram.

**Draw the following network diagrams**

1. **Activity Preceding activity**

A -

B -

C A

D C

E B & C

A

B

C

D

E

1. **Activity Preceding activity**

A -

B -

C A

D A

E B

F B

G C&E

H C,E, & F

A

D

E

C

B

G

H

F

1. Preceding activities are in brackets:

A

E

D

C

B

G

H

F

I

J

K

A(-)

B(-)

C(A)

D(B)

E(A&C)

F(B)

G(C&D)

H(G&F)

I(E)

J(H&I)

K(J)

**TIME ESTIMATES IN NETWORK ANALYSIS**

Time analysis of the network is useful for planning the various activities of a project. Time analysis requires an estimation of time to complete an activity. An activity time is the forecast of time expected of each activity from the starting time to its completion under normal conditions. The basic objective of time analysis is to get a planned schedule of the project. The plan should include the following:

1. Total completion time for the project
2. Earliest time when each activity can begin
3. The latest time when each activity can be started without delaying the total project.
4. Float for each activity – that is the amount of time by which the completion of an activity can be delayed without delaying the total project completion
5. Identification of critical activities and critical path

The analysis of the projects time can be achieved by using:

1. Single time estimates of each activity
2. Multiple time estimates of each activity

**CRITICAL PATH METHOD (CPM)**

If the times of all activities of a particular project are given, the time taken for each of the possible paths through the network can be evaluated.

**Role of CPM in project planning and coordination (importance)**

1. The network indicates the specific activities required to complete a project.
2. The network indicates the interdependence and sequence of specific activities.
3. Indicates start and finish time of each activity of the project.
4. It indicates the critical path and its duration of time.

**Assumptions of CPM**

1. A project can be subdivided into a set of predictable independent activities.
2. The precedence relationship of project activities can be completely represented by a non-cyclical network graph in which each activity connects directly with the immediate successors.
3. Activity times may be estimated as single point estimates or as 3 point estimates (that is, optimistic, pessimistic, and most likely) and are independent of each other.

**Key concepts**

1. **Earliest start time (EST)** – this is the earliest time at which an activity can begin. It is the earliest time at which a succeeding activity can start.
2. **Latest start time (LST)** – refers to the earliest expected time – the latest time which an activity can begin without affecting the normal project duration. The latest possible time at which a preceding activity can finish without delaying project duration.
3. **Earliest finish time (EFT) –** earliest time at which an activity can be completed.
4. **Latest finish time (LFT) –** the latest time at which an activity can be finished without affecting the normal time duration.
5. **Slack –** the amount of time by which the start of an activity may be delayed without affecting the overall duration of a project. Free or spare time in the network.
6. **Critical path –** the path in the network with the longest time.

2 features to note:

1. There can be no more than one critical path in a network.
2. To reduce the duration of a project requires shortening the time of an activity that is on the critical path.

**COMPUTATION OF TIME ESTIMATES**

There are three main categories:

1. Forward pass (associated with EST)
2. Backward pass (LST)
3. Critical path

**Illustration**

**Activity Pre-requisite Time**

A - 1

B A 2

C A 3

D B 4

E C 1

F D&E 2

A

E

D

C

B

F

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | **A** | **B** | **C** | **D** | **E** | **F** |
| **EST** | 0 | 1 | 1 | 3 | 4 | 7 |
| **LST** | 0 | 1 | 1 | 3 | 6 | 7 |

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | **A** | **B** | **C** | **D** | **E** | **F** |
| **EFT** | 1 | 3 | 4 | 7 | 7 | 9 |
| **LFT** | 1 | 3 | 6 | 7 | 7 | 9 |

**Identification of critical path**

A critical path comprises of critical activities. An activity is critical when:

1. EST = LST of tail event
2. EST = LST of head event
3. EST of head event – EST of tail event = time duration of the activity

Critical path: A – B – C – F

To denote the critical path use || or a double-arrow.

**Project duration**

Project duration is the sum of the time taken by the critical activities on the critical.

Project duration = 1 + 2 + 4 + 2 = 9 days

**Exercise 1**

Precedence table for a project is given as follows:

|  |  |  |
| --- | --- | --- |
| **Activity** | **Predecessor** | **Time duration** |
| **A** | - | 3 |
| **B** | - | 2 |
| **C** | - | 2 |
| **D** | A | 4 |
| **E** | B | 4 |
| **F** | B | 7 |
| **G** | C | 4 |
| **H** | A, D | 2 |
| **I** | B, E | 5 |
| **J** | B, C, F, G | 6 |
| **K** | A, B, D, E, I, H | 3 |

1. Draw the network diagram
2. Determine EST, LST, EFT & LFT of each activity
3. Find the critical path
4. Find project duration

**Solution-**

A

D

H

B

E

G

F

C

I

K

J

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **A** | **B** | **C** | **D** | **E** | **F** | **G** | **H** | **I** | **J** | **K** |
| **EST** | 0 | 0 | 0 | 3 | 2 | 2 | 2 | 7 | 6 | 9 | 11 |
| **LST** | 0 | 0 | 0 | 6 | 2 | 2 | 5 | 10 | 7 | 9 | 12 |
| **EFT** | 3 | 2 | 2 | 7 | 6 | 9 | 9 | 11 | 11 | 15 | 15 |
| **LFT** | 6 | 2 | 5 | 10 | 7 | 9 | 9 | 12 | 12 | 15 | 15 |

**Exercise 2**

A project consists of the following activities with duration n weeks of each activity given in brackets.

A(12), B(8), and C(14), can be executed concurrently. A and B precede D(4). B precedes E(2), F(10), and H(16). F and C precede G(6). E and H precede I(4) and J(8). C, F, and J precede K(4). K precedes L(8). D, I, G, and L are the terminating activities in the project.

1. Construct the precedence table, and hence draw the network for the project.
2. Determine the critical path and the project duration using the forward pass and backward pass method.
3. Construct a table of the EST, LST, EFT, LFT, and total float for each activity.

**FLOAT CALCULATIONS**

If an activity is not on the critical path then it is not possible to increase its duration without increasing the total project duration. The extra time is known as float. There are three types of float:

1. Total float
2. Free float
3. Independent float
4. **Total float**

It is the amount of spare time a path of activities could be delayed without affecting the overall project duration.

*Total float = Latest head time – Earliest tail time – Activity duration*

*TF = LFT – EST - tij*

Activity A*ij* is that activity with the tail end I and the head end j. Total float on critical activity is always zero.

**Interpretation of total float**

1. Negative total float – implies inadequate resources; activity may not finish on time
2. Zero total float – resources are just enough to complete the activity; activities are not delayed
3. Positive total float – excess resources; thus they can be reallocated to other activities or delay activities by amount of the total float.
4. **Free Float**

Amount of time an activity could be delayed when all preceding activities are completed as late as possible and all succeeding activities completed as early as possible.

*FF = EHT – ETT - tij*

*= EFT – EST - tij*

1. **Independent float**

Amount of time an activity could be delayed when all preceding activities are completed as late as possible and all succeeding activities completed as early as possible.

*IF = EHT – ETT – tij*

*= EFT – EST - tij*

Note:

* EHT = EFT
* LTT = LST

Slacks: free or spare time in a network. Slack refers to events while float focuses on activity.

Event Slack (ES)

*ES = Latest event time – Earliest event time*

**2 types of slack:**

* Head slack (HS) = Lj – Ej
* Tail slack (TS) = Li – Ei

In terms of slack events:

TF = Lj – Ei *– tij*

FF = TF – HS

IF = FF – TS

[Get EST, LST, EFT, LFT, TF, FF, IF, HS, and TS for exercise 1]

**PROJECT EVALUATION AND REVIEW TECHNIQUE (PERT)**

Methodology:

1. Project planning – drawing the network diagram
2. Time estimation
3. Scheduling – computation of the latest and earliest allowable start and finish time. Hence we determine the critical path, the float and the slacks.
4. Time cost trade-offs – to determine the cost of reducing the completion time of the project. The cost of reducing the project-completion-time as well as time-cost trade-offs of activity performance times are put into account for activities on the critical path.
5. Resource allocation – the feasibility of each schedule must be checked with respect to man-power and equipment requirement.
6. Project control – the project is controlled by checking the progress against the schedule, assigning and scheduling man-power and equipment, and analyzing the effects of delays.

**PERT system of 3 estimates**

1. Optimistic time (O, a, or t0)
2. Most likely time (M, tm or ML)
3. Pessimistic time (P, b, or tp)
4. **Optimistic time –** It is the estimate of the shortest minimum possible time an activity can take to be completed. This is usually under ideal conditions.
5. **Pessimistic time –** It is the longest possible time that an activity can take.
6. **Most likely time –** the time which the activity will take most frequently if performed a number of times.

The three time estimates O, M and P are combined statistically to develop the expected time (te) for an activity.

**2 main assumptions:**

1. The three time estimates O, M and P form the end-points and the mode of beta distribution. Both optimistic and pessimist time are equally likely to occur whereas the probability of occurrence of most likely time is 4 times that of O and P.

O

M

te

P

1. The probability of O:M:P = 1:4:1





Deviation/Variance

=*standard deviation squared*

Standard deviation



The standard deviation of the project = 

**Procedure for calculating time estimates using PERT**

1. Prepare a table of expected duration, the variance for each of the activities of the project.
2. Draw the project network diagram of activities based on the expected duration of activities.
3. Find or determine the critical path.
4. Find the total expected duration of the project based on the network also referred to as the average duration of the project.
5. Standard deviation of the entire project
6. Find the probability of completing the project within a particular time period.

**Example**

A project has the following activities and characteristics.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Activity** | **Optimistic**  **time** | **Most likely**  **time** | **Pessimistic**  **time** | **Expected**  **time**  **duration (te)** | **Variances** |
| 1-2 | 2 | 5 | 8 | 5 | 1 |
| 1-3 | 4 | 10 | 16 | 10 | 4 |
| 1-4 | 1 | 7 | 13 | 7 | 4 |
| 2-5 | 5 | 8 | 11 | 8 | 1 |
| 3-5 | 2 | 8 | 14 | 8 | 4 |
| 4-6 | 6 | 9 | 12 | 9 | 1 |
| 5-6 | 4 | 7 | 10 | 7 | 1 |

5

10

8

8

7

9

7

*Critical path: 1-3-5-6*

*Total duration time = 10 + 8 + 7 =25 days*

Standard deviation of the project



**COST ANALYSIS - Costs and Networks**

Concerned with the cost of activities and of the project as a whole. The main objective is to calculate the cost of various activity durations and find the cheapest way of reducing the overall project duration.

**Concepts associated with PERT costs**

1. Normal cost – cost associated with the normal time estimate for an activity.
2. Crash cost – cost associated with the minimum possible time for an activity.
3. Crash time – minimum possible time for an activity.
4. Cost slope – average cost of shortening an activity by 1 time unit.



**Example**

The normal time of an activity is 12 days at a cost of ₤ 480 and a crash time of 8 days at ₤ 640.

= ₤ 40 per day

1. Least cost scheduling or crashing – the process of finding the least cost method of reducing the overall project duration from time period to time-period.

Note: only critical activities affect the project duration and thus they are the only ones whose time can be reduced (or crashed).

**Example**

A project has the following characteristics:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Activity | Pre-requisite | Time | | Cost | | Cost  Slope |
|  |  | Normal | Crash | Normal | Crash |
| A | - | 5 | 3 | 500 | 620 | 60 |
| B | - | 4 | 2 | 300 | 390 | 45 |
| C | A | 7 | 6 | 650 | 680 | 30 |
| D | A | 3 | 2 | 400 | 450 | 50 |
| E | B, C | 5 | 3 | 850 | 1000 | 75 |

Network Diagram

A

5

B

4

D

3

E

5

*Critical path: ACE*

*Total normal cost = ₤ 2700*

*Time duration 17 days*

Time scheduling

Begin by reducing by 1 day activities on the critical path with the lowest cost slope i.e. reduce activity C by 1 day at an extra cost of ₤ 30.

*Total normal cost is ₤ 2700 with 17 days*

*16 day duration (C) at a total cost of 2730*

*15 day duration (A) at a total cost of 2790*

*14 day duration**(A) at a total cost of 2850*

*13 day duration (E) at a total cost of 2925*

*12 day duration (E) at a total cost of 3000*

The project can be completed in 12 days at a cost of ₤ 3000.

**Exercise**

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Activity | Pre-requisite | O | M | P | Time | | Cost | |
|  |  |  |  |  | Normal | Crash | Normal | Crash | Cost Slope |
| A | - | 3 | 4 | 5 | 4 | 3 | 400 | 600 | 200 |
| B | - | 6 | 8 | 10 | 8 | 7 | 500 | 700 | 200 |
| C | F, D | 2 | 3 | 4 | 3 | 2 | 900 | 1000 | 100 |
| D | B | 4 | 5 | 12 | 6 | 5 | 600 | 700 | 100 |
| E | - | 5 | 7 | 9 | 7 | 6 | 800 | 900 | 100 |
| F | A | 9 | 16 | 17 | 15 | 14 | 950 | 1250 | 300 |
| G | B | 8 | 12 | 16 | 12 | 10 | 670 | 870 | 100 |
| H | G | 8 | 10 | 12 | 10 | 8 | 890 | 990 | 50 |
| J | L | 4 | 5 | 6 | 5 | 3 | 1000 | 1200 | 100 |
| K | E | 7 | 8 | 15 | 9 | 7 | 400 | 600 | 100 |
| L | G, K | 8 | 11 | 14 | 11 | 9 | 1200 | 1400 | 100 |

Required:

1. Expected time duration for each activity using PERT formula (te).
2. Network diagram.
3. Critical path and total time for the project.
4. Standard deviation of the project.
5. Least cost schedule – the total time and cost for the project.

**17. QUEUING THEORY**

A queue refers to items awaiting service. Queues may include people, cars, telephone calls, and aircrafts queuing for landing. Queuing theory is a construction of mathematical models of various types of queuing systems so that predictions may be made about the system. Queues form when the rate of arrival of items requiring service is greater than the rate of service.

**Concepts**

1. **Arrival –** the element concerned with how items arrive in the system.
2. **Queue –** what happens between the arrival of the item and the time when the service is carried out. It is known as queue discipline, which is generally assumed to be first-come-first-serve.
3. **Service –** time taken to serve a customer.
4. **Outlet –** exit from the system.
5. **Queuing –** the whole situation considered from arrival to exit. The time in the system is generally taken to be the queuing time plus the service time.
6. **Customer –** items awaiting service: people, machines.
7. **Service station –** the point where service is provided.
8. **Waiting time –** the time a customer spends in the queue before being served.
9. **Time spent by customer in the system –** waiting time + service time.
10. **No.** **of customers in the system –** no. of customers in the queue + no. of customers being served.
11. **Queue length –** no. of customers waiting in the queue.
12. **Queuing system –** consists of the arrival of customers waiting in the queue being picked for the service, customers being served and the departure of customers.

**Types of Queuing Systems**

1. Single queue – single serving point e.g. hospitals
2. multiple queues – multiple serving points e.g. banks, supermarkets
3. single queue – multiple service point e.g. banks
4. multiple queues – single serving point e.g. telephone calls

**Assumptions/Characteristics of a single channel queuing model**

1. A single queue and a single service point.
2. First come – first served queuing discipline.
3. The queue has infinite capacity (i.e. waiting space available for customers)
4. The arrivals are random and follow a Poisson distribution.
5. No simultaneous arrival in a small time interval
6. Service times are random and follow a negative exponential distribution.
7. Traffic intensity is less than 1.

**BENEFITS OF QUIEUING THEORY**

1. Scheduling of aircraft at landing and take-off from busy airports.
2. Scheduling of mechanical transport fleets.
3. Scheduling of work and jobs in production control.
4. Minimization of congestion due to traffic delay.
5. Scheduling of parts and components to assembly lines.
6. Inventory analysis and inventory control.

**SINGLE CHANNEL QUIEUING MODEL**

**Arrival and service rates**

There are 2 major factors in the queuing problems – the arrival and the service rates.

1. **Arrival rate**

This is the average rate of arrivals per unit of time. It is denoted by  e.g. 20 students on average visit the library each hour – 20 items per unit of each hour



On average a plane arrives at the airport every 5 minutes.



1. **Service rate(****)**

The average no. of services completed in a unit of time.



e.g. a cashier at a supermarket serves on average 480 customers per 8 hour shifts



1. **Traffic intensity (****)**

The ratio of average arrival rate to the average service rate.



Alternatively ewe can get traffic intensity by dividing average service time by average time between arrivals



In a single queue people on average arrive at the rate of 15 per hour whereas 20 of them are served per hour. Find the traffic intensity.



Alternative method:







Assumption – the mean arrival rate is less than the mean service rate. . Under single queuing this queue will come to an end.

**Interpretation of traffic intensity (****)**

1. If traffic intensity is zero then there is no queue.
2. If traffic intensity = 1 or more the queue will theoretically be of infinite length, it will be continuous.
3. Traffic intensity is the probability that the service point is busy.

**QUEUING FORMULAE**

1. **No. of customers**
2. The average no. of customers /items in the queue when there is a queue



1. Average no. of customers or items in the system:

Items in the queue plus the customers being served



1. **Time taken by customers/items**
2. Average time in the queue (Tq)



1. Average time taken in the system (Ts)



1. **Probabilities**
2. The probability of queuing on arrival i.e. the probability that the service point is busy = traffic intensity.



1. The probability of there being no queue on arrival



1. The probability of there being one or more items in the system



**Example 1**

At a currency exchange bureau, on average a customer arrives every 5 minutes and takes 4 minutes to be served. Considering the assumptions of a single channel queuing model, determine the following:

1. Average no. of arrivals per minute ()
2. Service rate()
3. The traffic intensity
4. Fraction of the time the service point/cashier is busy
5. The probability the cashier is busy
6. Expected no. of customers in the system
7. The average length of the queue (no. of customers waiting in the queue)
8. The mean time a customer spends in the system
9. The mean time a customer spends in the queue

**Solution**

1. Arrivals per minute



1. Service rate



1. Traffic intensity



1. Fraction of time cashier is busy

80% of the time

1. Probability that cashier is busy = 0.8
2. Expected no. of customers in the system



1. Average length of the queue



1. The mean time a customer spends in the system



1. The mean time a customer spends in the queue



**Exercise**

A mini supermarket has a cashier who serves 48 customers per hour on average during the rush hour. The customers arrive at the rate of 40 customers per hour. Assuming a single channel queuing model, determine:

1. The probability the cashier is idle
2. The average no. of customers in the queuing system
3. The average time a customer spends in the system
4. The average no. of customers in the queue
5. The average time a customer spends in the queue waiting for the service

**Solution**





1. Probability cashier is idle



1. Average no. of customers in the queuing system



1. Average time customers spend in the system



1. Average no. of customers in the queue



1. Average time a customer spends in the queue waiting to be served

